

The group of autoequivalences and the Fourier-Mukai number of a projective manifold

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Abstract

Let X be a smooth projective variety and $\text{Aut}(D(X))$ the group of autoequivalences of the derived category of X . In this paper we show that X has no Fourier-Mukai partner other than X when $\text{Aut}(D(X))$ is generated by shifts, automorphisms and tensor products of line bundles.

1 Introduction

In this paper, a *projective manifold* means a smooth projective variety over the complex number field \mathbb{C} . We consider the derived category $D(X)$ of a projective manifold X . That is $D(X)$ is the bounded derived category of the abelian category $\text{Coh}(X)$ of coherent sheaves on X . As is well-known since [Muk], for some X , there is another projective manifold Y which is not isomorphic to X but the derived category $D(Y)$ is equivalent to $D(X)$ as triangulated categories. We call such a Y a *Fourier-Mukai partner* of X .

Let $\text{FM}(X)$ be the set of isomorphic classes of Fourier-Mukai partners of X . It is conjectured in [Kaw] that the set $\text{FM}(X)$ is a finite set. For instance if X is an algebraic surface, the conjecture holds (cf. [BM]). Hence we call the cardinality of $\text{FM}(X)$ the *Fourier-Mukai number* of X . We note that, when X is a projective K3 surface, [HLOY] makes the counting formula of the Fourier-Mukai number of X .

Let $\text{Aut}(D(X))$ be the group of autoequivalences of $D(X)$. There are a few cases when the group $\text{Aut}(D(X))$ is exactly determined. For instance $\text{Aut}(D(X))$ is determined by [BO, Theorem 3.1] when the canonical bundle K_X (or $-K_X$) is ample. When X is an abelian variety, $\text{Aut}(D(X))$ is determined by [Orl2].

In this paper we shall study the relation between $\text{Aut}(D(X))$ and $\text{FM}(X)$:

Does $\text{Aut}(D(X))$ give us informations on $\text{FM}(X)$?

We shall give an answer to this question in an easy case. Namely our theorem is the following:

Theorem 1.1. (*= Theorem 3.2*) *Let X be a projective manifold. We assume that $\text{Aut}(D(X))$ is trivially generated (cf. Definition 2.1). Then $\text{FM}(X) = \{X\}$.*

As an application of Theorem 1.1, we show the following:

Corollary 1.2. (*= Corollary 3.5*) *Let X be a projective manifold such that $\deg K_X|_C \neq 0$ for any irreducible curve $C \subset X$. Then $\text{FM}(X) = \{X\}$.*

We found the paper [Fav] and noticed that Theorem 1.1 is a special case of [Fav, Corollary 4.3] on arXiv after we have finished this paper. However our proof is independent of Favero's proof and much simpler than his. In addition our motivation is essentially different from his.

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2 The group of autoequivalences

In this section we recall some known results on $\text{Aut}(D(X))$. First we give the following easy examples of autoequivalences:

1. The shift of complexes $[1] : D(X) \rightarrow D(X)$.
2. The (right derived) functor $\mathbb{R}f_* = f_* : D(X) \rightarrow D(X)$, where f is an automorphism of X .
3. The tensor products by $L \otimes : D(X) \rightarrow D(X)$ where $L \in \text{Pic}(X)$ and $\text{Pic}(X)$ is the Picard group of X .

Definition 2.1. We define the subgroup $\text{Tri}(X)$ of $\text{Aut}(D(X))$ by the following condition: $\text{Tri}(X)$ is the subgroup generated by shifts, automorphisms and tensor products with line bundles. $\text{Tri}(X)$ is called the *trivial group generated by X* . If $\text{Aut}(D(X)) = \text{Tri}(X)$, $\text{Aut}(D(X))$ is said to be *trivially generated*.

Remark 2.2. The trivial group $\text{Tri}(X)$ is written by the following:

$$\text{Tri}(X) = (\text{Aut}(X) \ltimes \text{Pic}(X)) \times \mathbb{Z}[1],$$

where $\text{Aut}(X)$ is the group of automorphisms of X . Namely for $f \in \text{Aut}(X)$ and $L \in \text{Pic}(X)$, we have

$$f_* \circ L \otimes \circ f_*^{-1}(?) = f_*(L) \otimes (?).$$

For instance, when K_X or $-K_X$ is ample, $\text{Aut}(D(X))$ is trivially generated by [BO, Theorem 3.1]. The first nontrivial example of an autoequivalence was found by Mukai. Let us recall his example.

Let A be an abelian variety, \hat{A} the dual abelian variety of A and \mathcal{P} the Poincaré line bundle on $A \times \hat{A}$. We define the functor $\Phi : D(\hat{A}) \rightarrow D(A)$ by the following way:

$$\Phi : D(\hat{A}) \rightarrow D(A), \quad \Phi(?) := \mathbb{R}\pi_{A*}(\mathcal{P} \otimes \pi_{\hat{A}}^*(?)), \quad (2.1)$$

where $\pi_A : A \times \hat{A} \rightarrow A$ and $\pi_{\hat{A}} : A \times \hat{A} \rightarrow \hat{A}$ are the natural projections. Then Φ is an equivalence between $D(\hat{A})$ and $D(A)$ by [Muk, Theorem 2.2]. The definition (2.1) seems special, but the following theorem claims that it is sufficiently general.

Theorem 2.3. ([Orl, Theorem 2.18]) *Let X be a projective manifold and Y a Fourier-Mukai partner of X . Then, for any equivalence $\Phi : D(X) \rightarrow D(Y)$, there is an object $\mathcal{P}^\bullet \in D(X \times Y)$ such that*

$$\Phi(?) = \mathbb{R}\pi_{Y*}(\mathcal{P}^\bullet \overset{\mathbb{L}}{\otimes} \mathbb{R}\pi_X^*(?)),$$

where π_X (resp. π_Y) is the natural projection from $X \times Y$ to X (resp. Y). Moreover \mathcal{P}^\bullet is unique up to isomorphism.

Thus we obtain the following useful corollary:

Corollary 2.4. *Let x_0 be a closed point of X and \mathcal{O}_{x_0} the skyscraper sheaf of x_0 . If $\Phi(\mathcal{O}_{x_0}) \simeq \mathcal{O}_{y_0}$ for a closed point $y_0 \in Y$, then there is a Zariski open subset U of X such that*

$$x \in U \Rightarrow \exists y \in Y \text{ such that } \Phi(\mathcal{O}_x) \simeq \mathcal{O}_y.$$

In addition, assume that for all $x \in X$ there is a closed point $y \in Y$ such that $\Phi(\mathcal{O}_x) \simeq \mathcal{O}_y$. Then there is an isomorphism $f : X \rightarrow Y$ and $L \in \text{Pic}(Y)$ such that $\Phi(?) \simeq L \otimes (f_(?))$.*

Proof. See [Huy, Corollary 5.23 and Corollary 6.14]. □

3 Proof of Theorem 1.1

In this section we shall prove our main theorem. We first cite a key lemma of the proof essentially due to [BO]. We define the support $\text{Supp}(E)$ of $E \in D(X)$ by

$$\text{Supp}(E) = \bigcup_i \text{Supp}(H^i(E)),$$

where $H^i(E)$ is the i -th cohomology with respect to the t-structure $\text{Coh}(X)$.

Lemma 3.1. (*[BO, Proposition 2.2] or [Huy, Lemma 4.5]*) *Let X be a projective manifold and $E \in D(X)$. Assume that*

$$\dim \text{Supp}(E) = 0 \text{ and } \text{Hom}_{D(X)}(E, E[i]) = \begin{cases} 0 & (i < 0) \\ \mathbb{C} & (i = 0). \end{cases}$$

Then E is isomorphic to $\mathcal{O}_x[n]$ for some $x \in X$ and $n \in \mathbb{Z}$.

Lemma 3.1 is essentially due to [BO]. The above formulation of the lemma is due to [Huy].

Now let X and Y be projective manifolds and $\Phi : D(X) \rightarrow D(Y)$ an equivalence. Then we remark that Φ induces the natural group isomorphism $\Phi_* : \text{Aut}(D(X)) \rightarrow \text{Aut}(D(Y))$ by the following way:

$$\Phi_* : \text{Aut}(D(X)) \rightarrow \text{Aut}(D(Y)), \quad \Phi_*(F) := \Phi \circ F \circ \Phi^{-1}.$$

Theorem 3.2. (*=Theorem 1.1*) *Let X be a projective manifold. Assume that $\text{Aut}(D(X))$ is trivially generated. Then $\text{FM}(X) = \{X\}$.*

Proof. Let Y be an arbitrary Fourier-Mukai partner of X and $\Phi : D(X) \rightarrow D(Y)$ an equivalence. We fix Φ . We would like to show that Y is isomorphic to X .

Choose a very ample line bundle L_Y on Y and fix it. Since the induced morphism $\Phi_* : \text{Aut}(D(X)) \rightarrow \text{Aut}(D(Y))$ is an isomorphism, we have

$$\exists F \in \text{Aut}(D(X)) \text{ s.t. } \Phi_*(F) = L_Y \otimes (-).$$

We remark that the following diagram commutes:

$$\begin{array}{ccc} D(X) & \xrightarrow{\Phi} & D(Y) \\ F \downarrow & & \downarrow L_Y \otimes (-) \\ D(X) & \xrightarrow{\Phi} & D(Y). \end{array}$$

Since kL_Y has a global section for any positive integer $k \in \mathbb{Z}_{>0}$, we can make a morphism $E \rightarrow E \otimes kL_Y$ for any $E \in D(Y)$. Thus, for any $k \in \mathbb{Z}_{>0}$ and $E \in D(Y)$, we have $\text{Hom}_{D(Y)}(E, E \otimes kL_Y) \neq 0$. Thus we have

$$\text{Hom}_{D(X)}(\mathcal{O}_x, F^k(\mathcal{O}_x)) \cong \text{Hom}_{D(Y)}(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_x) \otimes kL_Y) \neq 0.$$

As $\text{Aut}(D(X))$ is trivially generated, F should be written by

$$F(?) = L_X \otimes f_*(?)[n],$$

where $f \in \text{Aut}(X)$, $L_X \in \text{Pic}(X)$ and $n \in \mathbb{Z}$. We shall prove that $n = 0$ and $f = \text{id}_X$.

Suppose to the contrary that $n \neq 0$. Since $n \neq 0$, for sufficiently large $\ell \in \mathbb{Z}_{>0}$, we have $\text{Hom}_{D(X)}(\mathcal{O}_x, F^\ell(\mathcal{O}_x)) = 0$ where F^ℓ is the ℓ times composition of F . This is contradiction. Hence n should be 0.

We assume that $f \neq \text{id}_X$. Then there is a closed point $x \in X$ such that $f(x) \neq x$. Since $F(\mathcal{O}_x) = \mathcal{O}_{f(x)}$, we have

$$\text{Hom}_{D(X)}(\mathcal{O}_x, F(\mathcal{O}_x)) = 0.$$

This is contradiction.

Thus we have $F = L_X \otimes (-)$. Hence for any positive integer k ,

$$\Phi(\mathcal{O}_x) \otimes kL_Y = \Phi(\mathcal{O}_x \otimes kL_X) = \Phi(\mathcal{O}_x).$$

Thus each Hilbert polynomial of $H^i(\Phi(\mathcal{O}_x))$ with respect to L_Y is constant. Since L_Y is very ample, it follows that $\dim \text{Supp}(H^i(\Phi(\mathcal{O}_x))) = 0$. Thus $\dim \text{Supp}(\Phi(\mathcal{O}_x)) = 0$. By Lemma 3.1, we have

$$\Phi(\mathcal{O}_x) = \mathcal{O}_{y_x}[n_x],$$

for some $y_x \in Y$ and $n_x \in \mathbb{Z}$. By the first half assertion of Corollary 2.4, n_x is locally constant. Hence, n_x is constant. So we put $n_x = n$. Then Y is isomorphic to X by the last half assertion of Corollary 2.4. \square

Remark 3.3. The converse of Theorem 3.2 does not hold. For instance, there are projective K3 surfaces X with Fourier-Mukai number one (See [HLOY] or [Ogu]). On the other hand, as is well-known by [ST], the spherical twist T_S by a spherical object¹ $S \in D(X)$ gives an autoequivalence of $D(X)$ which does not belong to $\text{Tri}(X)$. Thus $\text{Tri}(X)$ is a proper subgroup of $\text{Aut}(D(X))$.

¹For example a line bundle on X is a spherical object.

Let us consider the following three statements for a projective manifold X :

- (A) The canonical bundle K_X (or $-K_X$) is ample.
- (B) The autoequivalence group $\text{Aut}(D(X))$ is trivially generated.
- (C) The Fourier-Mukai number of X is one.

[BO] proved that $(A) \Rightarrow (B)$ and $(A) \Rightarrow (C)$. As we wrote in Remark 3.3, the converse does not hold. Our theorem claims that $(B) \Rightarrow (C)$.

Now we would like to show that the proposition $(B) \Rightarrow (A)$ does not hold.

Proposition 3.4. *Let X be a projective manifold such that $\deg K_X|_C \neq 0$ for any irreducible curve $C \subset X$. Then $\text{Aut}(D(X))$ is trivially generated.*

For instance, let Y be a projective manifold such that K_Y is ample and let $X \rightarrow Y$ be the blowing up at a point of Y . Then X satisfies the assumption.

Proof. We choose an arbitrary autoequivalence $F \in \text{Aut}(D(X))$ and fix it. Since the functor $\otimes K_X[\dim X]$ is the Serre functor, the following diagram commutes up to isomorphisms:

$$\begin{array}{ccc} D(X) & \xrightarrow{F} & D(X) \\ \otimes K_X \downarrow & & \downarrow \otimes K_X \\ D(X) & \xrightarrow{F} & D(X). \end{array}$$

Thus we have

$$F(\mathcal{O}_x) \simeq F(\mathcal{O}_x) \otimes K_X. \quad (3.1)$$

It suffices to show that $\dim \text{Supp}(F(\mathcal{O}_x)) = 0$ by Lemma 3.1 and Corollary 2.4. Suppose to the contrary that $\dim \text{Supp}(F(\mathcal{O}_x)) > 0$. Then there is an irreducible curve C contained in $\text{Supp}(F(\mathcal{O}_x))$. In particular we assume that $C \subset \text{Supp}(H^i(F(\mathcal{O}_x)))$ for some $i \in \mathbb{Z}$. Now we put $\mathcal{F} = H^i(F(\mathcal{O}_x))|_C$. Notice that $\text{rank } \mathcal{F} > 0$. Thus we have

$$\deg \mathcal{F} \otimes K_X|_C - \deg \mathcal{F} = \text{rank } \mathcal{F} \cdot \deg K_X|_C \neq 0$$

On the other hand $\deg \mathcal{F} \otimes K_X|_C - \deg \mathcal{F}$ should be 0 by (3.1). This is contradiction. \square

The next corollary easily follows from Proposition 3.4 and Theorem 3.2.

Corollary 3.5. *Notations are being as above. Then $\text{FM}(X) = \{X\}$.*

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